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Now it must be supposed that we integrate along some path lying on the cone and given by an equation $Z = F(\theta)$. But the homogeneity principle prepares us to see $Z$ disappear from the integral, as indeed it does, the result of the transformation being $\int X^L Y^M Z^{-L-M-1} dT = a^{L+1} b^{M+1} c^{-L-M} \int \cos^L \theta \sin^M \theta \, d\theta$.

Thus every integral of the type in question is transformable into this one particular form. In the author's first example, the required transformation is

$$\cos \theta = \sqrt{\frac{7}{3}} \cdot x(2 + 5x^2)^{-1/2}.$$

It may be added that, if we possess the general reduction formula developed above, nothing is gained by this transformation from the point of view of integration. In fact the reduction formula is exactly the same before and after transformation.

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## RECENT PUBLICATIONS.

### REVIEWS.

*The Principles of Geometry.* By H. F. BAKER. Volume 1, Cambridge University Press, 1922. 182 pp. Price 12 shillings.

That prince of teachers, the late Jules Tannery, once wrote in the introduction to a book by one of his former pupils: "Un petit livre est rassurant."

When a book is not only short but well printed, with wide margins and in a flowing style of the King's English, the reassuring impression is much strengthened. But the reader who takes up the work before us under these pleasant impressions, with the idea that in spite of a lack of any special preparation in the way of familiarity with the subject matter and point of view, he is going in a few hours to reach the real substance, this reader will have a very prompt chance to guess again.

It will take a much greater convulsion than the late World War to stop the output of books on Projective Geometry, especially from English writers. Yet so far, in England, they have clung to the tradition of Cremona and Stephen Smith, taking the metrical definition of cross ratios as fundamental, regardless of the fact that Klein proved half a century ago that the cross ratio can be reached by purely descriptive processes, and that Continental and American writers have been doing this in their texts for a generation. Sooner or later some Englishman was bound to fall into line, and it is a subject for satisfaction that the first should be such a distinguished scholar as Baker. The reason for the delay is doubtless this, that English boys are drilled in a subject of which Americans have scarcely heard the infelicitous name, Geometrical Conics. The metrico-projective treatment flows from this in the most natural manner. Moreover the present writer does not say that he is actually writing a book on projective geometry, and he starts his second chapter with a discussion of the descriptive properties of a limited region. But as soon as he has strengthened this to the point where it can stand on its legs, he adjoins to the universe of discourse ideal, or, as he says, "postulated" points, until he has the whole projective domain once more available.

The book has one most unusual characteristic, informality. The writer says in the fourth line of the preface: "It (the book) assumes those relations of position for points, lines and planes which, furnished with a pencil, a ruler, some rods and some string, a student may learn by drawing diagrams and making models."

There is no question of the independence of axioms: the writer states what he presupposes in the following fashion (pp. 4, 5): "Through an arbitrary point there pass an infinite number of lines, of which one passes through any other arbitrary point. Thus a line is determined uniquely by any two points. The line contains an infinite number of points besides these two determining ones, and is determined by any two of these. Through any given line there pass an infinite number of planes, of which one passes through any point not lying on the given line. Thus a plane is determined uniquely by any three points which do not lie in line. The plane contains an infinite number of points besides the determining ones, and is determined by any three of these which do not lie in line. The plane entirely contains the line which is determined by any two of its points. Thus if two lines have a common point, and we take two other points, one on each of the lines, the plane determined by the three points contains both the lines, that is, two lines with a common point determine a plane, containing both lines. It is also true that two lines which are in the same plane have a common point. We have said that there is an infinite number of planes, all containing the points of a given line; it is also true that any two planes have common points lying on a line, or that any two planes intersect in a line."

This casual method of statement is entirely characteristic. The writer seems to have an ingrained dislike for type display. He will head a long paragraph "So and So's Theorem" and put the actual statement, with or without italics, at such point in the argument as pleases him. The usual assortment of labels, Axiom, Postulate, Assumption, Restriction, Theorem, he calmly ignores. At times one is tempted to resent this, and exclaim irritably: "Why in the name of common honesty doesn't the good man tell us clearly once for all what he assumes, and what he proves?"

Fair and softly, dear reader, fair and softly, you won't get anywhere by storming at him. If you realize that this is a thoroughly scholarly work, and read every word, you will find that everything necessary is there. If you mistake it for a newspaper, and try to read nothing but the headlines, that is not his fault.

Here is a case in point. On p. 11 we find the usual quadrilateral construction for the harmonic point-set. A wicked gleam will come into the eye of the malevolent reader, as he exclaims: "Aha, how do we know that this yields anything new? How do we know that the diagonals of a complete quadrilateral are not concurrent? Does the writer expect us to use pencils and rods and string?"

Not in the least; when he gets good and ready, namely three pages later, he calmly says: "But we shall, in accordance with a general assumption referred to in the beginning, assume provisionally that the point $D$ does not coincide with the point $C$."

The first chapter deals with the general projective geometry, where no mention is made of order, or any equivalent characteristic of the real domain. We have the usual concurrence, collinearity and coplanarity theorems, which the writer well describes as "theorems of incidence." A good deal of attention is paid to "related ranges" of points. This means two ranges that are in perspective with the same third range. It is shown that two ranges which are related to a third are related to one another. Thus, when two one-dimensional forms are the first and last in a long string of projections and intersections, they may be connected by a single piece of machinery with only two joints. Next, of course, comes the fundamental theorem of projective geometry, whereby the projective relation between two one-dimensional forms is completely determined by the fate of three elements. The writer turns this over in every possible way, showing that it is equivalent to any one of four different theorems. The one which he prefers, and which constitutes the central assumption for this whole chapter, is Pappus's theorem, or Pascal's, where the conic is two lines. The preference for this particular form of statement is characteristic of the present work. This discussion of the reduction of perspectivities and the fundamental theorem is very careful; truth compels us to add that it is not short; the two require twenty-one pages.

All writers on the foundations of geometry, sooner or later, draw on arithmetic for aid; the present writer does so at the point we have now reached. Large letters are used to denote points, small letters denote a system of numbers about which we make all of the usual assumptions for addition and multiplication, *except the commutative law for the latter.* If the point $P$ be defined as the same as the point $rP$, then the points of the line determined by two given points will be the system of points linearly dependent on them in the present symbolism, and in fact, it is easily shown that the present system of linear dependence completely covers the former theorems of incidence.

The correlation of arithmetical and geometrical operations is next developed, very satisfactorily, on the following lines:

(*a*) The relation between the numbers $r$ and $-r$. The points $A + rB$ and $A - rB$ shall be harmonic conjugates with regard to the points $A$ and $B$.

(*b*) The relation between the numbers $r$, $s$, and $r + s$. The point $B$ shall be one vertex of a complete quadrangle, $A + rB$ and $A + sB$ shall be the intersections of a transversal through $B$ with one pair of opposite sides, $A$ and $A + (r + s)B$ shall be the intersections of this transversal with the remaining pair of sides. It is easy to show that this construction is independent of the choice of the quadrangle, and that addition is commutative and associative.

(*c*) The relation between the numbers $r$, $s$, and $rs$. A transversal shall meet one pair of opposite sides of a complete quadrangle in $A$ and $B$, a second pair in $A + rB$ and $A + sB$, while it meets a third pair in $A + B$ and $A + rsB$. This will make $rs = sr$, when, and only when, the construction yields a unique result, and this unicity of result is shown, in turn, to depend on Pappus's theorem. We thus reach in a new form a result, first exhibited by Hilbert, namely, that the independence of Pappus's theorem from the other theorems of incidence may be shown by the existence of number systems where multiplication is not commutative, the calculus of quaternions being the classical example.

The second chapter of our book takes a fresh start, dealing with a much more restricted domain, namely, a limited region of real space. Peano's axiom which requires that a line that meets one side of a triangle, while it meets a second side produced, shall meet the third side, is very fundamental. This is really a most useful assumption for it accomplishes two distinct ends, it gives sufficient conditions for the intersection of coplanar lines, and for the projection of betweenness into betweenness. Betweenness is, in fact, a very fundamental relation in this sort of development, and we have at the proper moment a continuity axiom based, as usual, on a Dedekind cut among the points of a line-segment. With continuity we can do many things: prove Pappus's theorem, establish connection with the algebra of real numbers, etc. But our author does not remain content with his limited space but presently fills it out to the complete space of projective geometry, introducing ideal elements by a device first employed, we believe, by Pasch. Suppose, in fact, that in this limited space we have a system of lines of which at least one goes through every point, and of which each two are coplanar. An example of such a system is the assemblage of all lines through a point of the region. When the system is not so constituted, *i.e.*, when no two intersect in the region, then we say that they determine an ideal or postulated point which lies on all of them. Ideal lines and planes are then introduced in a similar fashion, and the whole of projective space is presently available. One rather wonders why Baker did things in just this way, why he did not follow in the real domain the plan already pursued in the general domain of assuming at the outset the intersection of coplanar lines, and postulating the laws of separation of four elements of a one-dimensional fundamental form. The explanation is, perhaps, to be found on p. 2: "In geometry, as applied to the external world, we cannot but be conscious for instance, in dealing with the points of a line, of the difference between the points which we regard as accessible, and those which we regard as the others." This would seem to mean that historically, and psychologically, geometry begins with real accessible points, and that it is natural to base the logical structure on these also. Yet the other method of approach is very neat.

The third chapter is in the nature of reconciliation of the two that preceded it. It is shown how the assumption of complex points, *i.e.*, points corresponding to complex numbers, rounds out and completes the real space of the second chapter. There is, here, a real obscurity arising from the writer's reluctance to state his assumption explicitly. We are told on p. 141 that Ch. III is a continuation of Ch. I. That suggests that the assumptions made in Ch. I and not those in Ch. II are valid. Then we read on p. 152: "Conversely, two spaces of a like number of dimensions, with a point-to-point correspondence of such a kind that to every incidence of elements in one space corresponds a precisely similar incidence of elements in the other, are necessarily related (projective) as we see by recalling the construction of related ranges."

This amounts to saying that every transformation that carries a point to a point and a plane to a plane, or a line to a line, is a projective transformation; true in the real domain, but untrue in the complex one. Witness the transformation which consists in interchanging conjugate imaginary elements.

The final section shows how complex geometry may be built out of real geometry by defining complex elements as groups of real ones. The first successful development of this kind was given by Von Staudt, his method being to define a complex point as an elliptic involution of real collinear points, together with a sense of description. Baker, however, follows a later method devised by Klein, where complex points are defined by cyclic projectivities among real ones. His treatment is based very directly on the exposition of Vahlen, and does not seem to us as clear as the original, but it is fair to say that he explicitly announces that all he says on this subject should be regarded as provisional.

What shall we say of the book as a whole? Is it as brilliant and stimulating as Hilbert's *Grundlagen der Geometrie?* No, it is not. Is it as searching and profound as Veblen and Young's *Projective Geometry?* No one could make such a claim. Is it as crisp and clear-cut as Vahlen's *Abstrakte Geometrie?* Such is certainly not the case. Does it afford as direct and natural an approach to the standard theorems as does Enriques' *Geometria Proiettiva?* Such is not our candid opinion. What then? We must judge a book finally by what the author sets out to do. We turn back to the preface to see what he expects of his book: "It seeks to set these relations (of points and lines and planes) in an ordered framework of deduction, gradually rendered comprehensive and precise enough to include all subsequent theory. To this end it puts aside at first most of the intricate details which make up the burden of what is generally called geometry. . . . Experience has shown that many students, especially of the class who look forward to becoming Engineers or Physicists, to whom the geometry of the usual text-books is tiresome, find such a course stimulating and easy, when the matter is properly presented to them. The mathematician who has followed such a course will find that he has no cause to think that the wrong things have been presented to him."

So the author wishes us to judge his book as an introduction, especially suited to those students who do not intend to specialize in mathematics. Our judgment must finally depend on the extent to which we agree with his educational theory. If he be correct in his opinion that the usual subject matter should be replaced by something more generalized, more abstract, capable of opening wider horizons, why then he deserves the great credit that should be awarded to every real pioneer in education. On the other hand, if his first assumption be essentially unsound, if truth be on the side of the older theory, which believed that the human mind proceeded naturally from the concrete to the abstract, and insisted that the natural approach to geometry is through the clean-cut, definite theorems of elementary metrics, why then we still welcome him as the first of his nation to write a scholarly text on a topic of recognized importance and value. May he speedily be followed by others, anxious to continue the good work.

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